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Progressive type II censored exponential data analysis: method comparison with breakdown voltage case study

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Abstract

This study presents a comparative analysis of different estimation methods for the parameter θ of the exponential distribution under progressive type II censoring. We compare maximum likelihood estimation with Bayesian approaches using squared error and Kullback-Leibler loss functions under different prior specifications. The theoretical developments presented are well established in the statistical literature; our contribution lies in the systematic empirical comparison of these methods. Through simulation studies and real data application, we examine the finite-sample behavior of these estimators to provide practical guidance for researchers. A real dataset from Lawless (1982) illustrates the application of these methods.

Key words: comparative study, exponential distribution, progressive type II censoring, Kullback-Leibler loss function, Bayesian estimation, maximum likelihood estimation.

1. Introduction

Life testing reliability studies in business, manufacturing, engineering and many other fields can be costly and time-consuming and the experimenter may not be able to get full data on the failure times for all experimental units. Multiple censoring techniques are used to reduce the time and the cost of testing. In a typical life testing experiment, n denotes the total number of items initially placed on test. The censoring techniques: types I and II are the first traditional techniques used. The experimenter tries to end the experiment at a predetermined time point, say T, in the case of type I censoring technique, but the experiment is completed when a certain number of failures are observed, say m $(m < n)$, in the case of type II censoring technique. One important development in the censoring techniques is the progressive type II censoring technique, which allows the experimenter to remove surviving units during the experiment. Many authors have been discussing progressive censorship. Balakrishnan and Aggarwala (2000), Balakrishnan and Cramer (2014) and Aggarwala (1996) are good sources for additional information.

A progressive type II censoring is carried out in the following way. Imagine n items are used in a life test. One or more surviving items may be arbitrarily eliminated from

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the life test at the time of each failure (censoring). The censoring takes place in m steps. The following information comes from a progressive type II censored sample: $\mathbf{x} = (x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n})$ with $(r_1, r_2, \ldots, r_m)$ censoring scheme, where $x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}$ denotes the m recorded failure times, and $r_1, r_2, \ldots, r_m$ denotes the number of items removed from the life test at each failure time.

In recent years, several authors have discussed the estimation methods of various lifetime distributions when progressive type II censored data are used. Recent developments within the last five years include Dey et al. (2021), who analyzed progressive Type-II censored gamma distribution using computational approaches and investigated the performance of different estimators; Alshenawy et al. (2021), who developed progressive Type-II censoring schemes for extended odd Weibull exponential distribution with applications in medicine and engineering; Wu and Gui (2021), who proposed Bayesian estimation methods for Nadarajah-Haghighi distribution under progressive Type-II censoring with applications in reliability analysis; Almetwally et al. (2022), who analyzed progressive Type-II censoring for unit-Weibull distribution with optimal scheme and real data applications in reliability engineering; and Ren and Hu (2023), who developed estimation methods for inverse Weibull distribution under progressive Type-II censoring scheme using maximum likelihood, Bayesian, and inverse moment estimation approaches. Earlier foundation works include: Kundu and Raqab (2012), Pradhan and Kundu (2009) and Kim et al. (2011). Progressive type II censored sampling is a crucial sampling technique for collecting data in lifetime research.

Let $x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}$ represent the failure times of m independent and identically distributed random variables from an exponential distribution with probability density function (pdf)

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x} & \text{if } x > 0, \theta > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{1}$$

and the cumulative distribution function

$$F(x;\theta) = 1 - e^{-\theta x}, \quad x > 0, \theta > 0. \tag{2}$$

The reliability and hazard functions of the exponential distribution are respectively given by

$$r(t) = e^{-\theta t} \quad \text{and} \quad h(t) = \theta$$

where $t > 0$, $\theta > 0$.

It is important to note that for the exponential distribution, the parameter θ is identical to the hazard function, i.e., $h(t) = \theta$ (constant hazard rate). This fundamental property will be acknowledged throughout our analysis.

The exponential distribution is widely recognized as one of the most fundamental probability distributions in statistical theory and practice (Lawless (1982); Balakrishnan and Aggarwala (2000)). Recent applications demonstrate the utility of exponential distribution in modeling electronic component lifetimes and mechanical system failure rates in reliability engineering (Rasheed (2023)). Sapkota et al. (2025) utilize the exponential distribution as a base to construct a novel "New Odd-type Exponential Distribution" within a flexible family, demonstrating its superior performance in bias reduction, model selection, and goodness-of-fit tests on engineering failure times and medical survival datasets.

The theoretical methods for maximum likelihood and Bayesian estimation under progressive censoring are well established in the statistical literature. Abufoudeh et al. (2019) and Abu Awwad et al. (2019) have studied Bayesian estimation under Kullback-Leibler loss for exponential distributions, while the maximum likelihood approach has been extensively covered by Balakrishnan and Aggarwala (2000) and others.

Based on the progressive type II sample $\mathbf{x} = (x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n})$ of size m, the objective of this article is to provide a comparative analysis of different estimation methods for the unknown parameter, reliability and hazard functions of the exponential distribution under the Kullback-Leibler loss and squared error loss functions. Our contribution is methodological and empirical rather than theoretical, focusing on the practical comparison of these well-established methods.

The squared error loss function (SELF) is defined as

$$L_{SE}(\theta,\hat{\theta}) = (\theta-\hat{\theta})^2$$

Following Kullback and Leibler (1951), who developed a divergence measure called the Kullback-Leibler divergence measure, this measure depends on the difference between the exact distribution $f(x|\theta)$ and the approximate distribution $\hat{f}(x|\hat{\theta})$ denoted by $KL(f,\hat{f})$ with property $KL(f,\hat{f}) \geq 0$. For further information, one can see Abufoudeh et al. (2019) and Abu Awwad et al. (2019). The Kullback-Leibler loss function is given by

$$KL(f,\hat{f}) = \int_{-\infty}^{\infty} f(x|\theta)\log\frac{f(x|\theta)}{\hat{f}(x|\hat{\theta})}dx = \frac{\hat{\theta}}{\theta} - \log\frac{\hat{\theta}}{\theta} - 1.$$

This Kullback-Leibler loss function (KELF) is denoted by $L_{KL}(\theta,\hat{\theta})$, where

$$L_{KL}(\theta,\hat{\theta}) = \frac{\hat{\theta}}{\theta} - \log\frac{\hat{\theta}}{\theta} - 1$$

The remainder of the article is structured as follows. Section 2 reviews the well-known method of deriving the maximum likelihood estimators of the unknown parameter θ of the exponential distribution. Since $h(t) = \theta$ for the exponential distribution, the estimator for the hazard function is identical to that for θ. Section 3 presents the standard Bayesian estimation methods under both SELF and KELF based on progressive type II censored data. Section 4 presents our main contribution: a comparative simulation study of the estimation results based on multiple progressive type II samples of different schemes, emphasizing that we are comparing different statistical paradigms rather than determining which is superior. Also, analysis of a real dataset from reliability studies is presented in Section 4. Section 5 presents the conclusions.

2. Frequentist method

This section reviews the standard maximum likelihood estimation approach for progressive type II censored exponential data, following the well-established methodology in the literature.

Suppose we observe a progressive type II censored sample $\mathbf{x} = (x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n})$ of size m from a total sample of size n, where the underlying distribution is exponential with parameter θ. As established by Balakrishnan and Aggarwala (2000), the likelihood function of progressive type II sample from exponential distribution is defined as

$$L(\theta|\mathbf{x}) = C\prod_{i=1}^{m} f(x_{i:m:n})[1-F(x_{i:m:n})]^{r_i}, \tag{3}$$

where $C = n(n-1-r_1)(n-2-r_1-r_2)\cdots(n-m+1-r_1-\cdots-r_{m-1})$.

By using (1), (2) and (3), we directly obtain

$$L(\theta|\mathbf{x}) = C\theta^m e^{-\theta\sum_{i=1}^{m}(1+r_i)x_{i:m:n}} \tag{4}$$

Using the logarithmic function for both sides in Eq. (4), we obtain the log-likelihood function

$$\ln L(\theta|\mathbf{x}) = \ln C + m\ln\theta - \theta\sum_{i=1}^{m}(1+r_i)x_{i:m:n}.$$

Setting the derivative of $\ln L(\theta|\mathbf{x})$ to zero produces the maximum likelihood estimator (MLE) of θ, which is the standard result:

$$\hat{\theta}_{MLE} = \frac{m}{\sum_{i=1}^{m}(1+r_i)x_{i:m:n}}. \tag{5}$$

By the invariance property of MLEs:

$$\hat{r}(t)_{MLE} = e^{-t\hat{\theta}_{MLE}} = e^{-\frac{mt}{\sum_{i=1}^{m}(1+r_i)x_{i:m:n}}} \tag{6}$$

Since $h(t) = \theta$ for the exponential distribution, we have $\hat{h}(t)_{MLE} = \hat{\theta}_{MLE}$. This eliminates the need for separate hazard function calculations.

3. Bayesian estimation and credible intervals

This section presents the standard Bayesian estimation procedures for exponential distributions under progressive censoring. The theoretical developments reviewed here are well established in the Bayesian literature.

Based on the observed m progressive type II sample $\mathbf{x} = (x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n})$, we apply standard Bayesian methodology to estimate the unknown parameter of the exponential distribution. Since the hazard function $h(t) = \theta$, estimating θ provides estimate for the hazard functions. KELF and SELF are used to obtain the point estimation of the unknown parameter θ and the reliability function.

3.1. Bayesian estimation for θ

Following standard Bayesian theory, we derive the Bayesian estimator of the unknown parameter θ under both squared error and Kullback-Leibler loss functions. The posterior distribution combines the information in the sample and the prior information, making it straightforward to use the Bayes technique. The posterior distribution of θ given $\mathbf{x} = (x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n})$ is obtained as

$$\pi(\theta|\mathbf{x}) = \frac{L(\theta|\mathbf{x})\pi_1(\theta|a,b)}{\int_0^\infty L(\theta|\mathbf{x})\pi_1(\theta|a,b)d\theta}. \tag{7}$$

If θ has a conjugate gamma prior $\pi_1(\theta|a,b)$ with hyper-parameters $a > 0$ and $b > 0$, then the pdf of θ given a and b is defined as

$$\pi_1(\theta|a,b) = \begin{cases} \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta} & \text{if } \theta > 0 \\ 0 & \text{if } \theta \leq 0 \end{cases} \tag{8}$$

By substituting Eqs. (3) and (8) in Eq. (7), using standard conjugate prior theory, we obtain

$$\pi(\theta|\mathbf{x}) = \frac{(b+\sum_{i=1}^m(1+r_i)x_{i:m:n})^{m+a}}{\Gamma(m+a)}\theta^{m+a-1}e^{-\theta(b+\sum_{i=1}^m(1+r_i)x_{i:m:n})}. \tag{9}$$

In other words, the posterior distribution of θ given $\mathbf{x}$ is Gamma$(m+a, b+\sum_{i=1}^m(1+r_i)x_{i:m:n})$.

The Bayes estimate of θ under the squared error loss function is the posterior mean:

$$\hat{\theta}_{BSE} = E_{posterior}(\theta|\mathbf{x}) = \frac{m+a}{b+\sum_{i=1}^m(1+r_i)x_{i:m:n}}. \tag{10}$$

The Bayes estimate of θ under the Kullback-Leibler loss function is obtained by minimizing the risk function as follows:

$$E_{posterior}(KL(\theta,\hat{\theta})) = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - \log\frac{\hat{\theta}}{\theta} - 1\right)\pi(\theta|\mathbf{x})d\theta.$$

Setting the derivative of $E_{posterior}(KL(\theta,\hat{\theta}))$ to zero produces

$$\int_0^\infty \left(\frac{1}{\theta} - \frac{1}{\hat{\theta}}\right)\pi(\theta|\mathbf{x})d\theta = 0.$$

Solving for $\hat{\theta}$ yields

$$\hat{\theta}_{BKL} = (E_{posterior}(\theta^{-1}|\mathbf{x}))^{-1}$$

For the gamma posterior distribution, this gives the Bayes estimate of θ under KELF:

$$\hat{\theta}_{BKL} = \frac{m+a-1}{b+\sum_{i=1}^m(1+r_i)x_{i:m:n}}. \tag{11}$$

3.2. Bayesian estimation for $r(t)$

We provide the Bayes estimate of the reliability function under squared error and Kullback-Leibler loss functions. The Bayes estimate of $r(t)$ under squared error loss is computed as follows:

$$\begin{aligned}\hat{r}(t)_{BSE} &= E_{posterior}(r(t)|\mathbf{x}) \\ &= E_{posterior}(e^{-\theta t}|\mathbf{x}) \\ &= \left(\frac{b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}}{t+b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}}\right)^{m+a}. \end{aligned} \quad (12)$$

The Bayes estimate of a general function of parameter θ, say $g(\theta)$ with respect to KELF is obtained as follows:

$$\hat{\theta}_{B_{KE}} = (E_{posterior}(g(\theta)^{-1}|\underset{\sim}{x}))^{-1}.$$

The Bayes estimate of $r(t)$ under Kullback-Leibler loss is computed as follows:

$$\begin{aligned}\hat{r}(t)_{BKE} &= (E_{posterior}((e^{-\theta t})^{-1}|\underset{\sim}{x}))^{-1} \\ &= \left(\frac{(-t+b+\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n})}{(b+\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n})}\right)^{m+a}. \end{aligned} \quad (13)$$

3.3. Credible intervals for θ and $r(t)$

Using standard results for gamma distributions, the credible intervals of θ are constructed as follows. Since θ has gamma posterior distribution, the $(1-\tau)100\%$ credible interval of θ, say (I_L, I_U), can be calculated by solving the equations:

$$P(I_L < \theta < \infty) = 1-\frac{\tau}{2}, \quad (14)$$

$$P(I_U < \theta < \infty) = \frac{\tau}{2}. \quad (15)$$

The $(1-\tau)100\%$ lower credible bound for θ is derived by solving Eq. (14) with respect to I_L, as follows:

$$\int\limits_{I_L}^{\infty} \frac{\left(b+\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}\right)^{m+a}}{\Gamma(m+a)}\theta^{m+a-1}e^{-\theta\left(b+\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}\right)}d\theta = 1-\frac{\tau}{2}.$$

By letting the substitution $u = \theta(b + \sum_{i=1}^{m}(1+r_i)x_{i:m:n})$ directly we get

$$\int\limits_{\left(b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}\right)I_L}^{\infty} u^{m+a-1}\, e^{-u}\, du = \left(1-\frac{\tau}{2}\right)\Gamma(m+a).$$

The incomplete gamma function $\Gamma(c,d)$, defined as

$$\Gamma(c,d) = \int\limits_{d}^{\infty} x^{c-1}e^{-x}\, dx,\, c>0,\, d>0,$$

is employed to derive the relation:

$$\Gamma\left(m+a,\, (b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n})\,I_L\right) = \left(1-\frac{\tau}{2}\right)\Gamma(m+a). \tag{16}$$

Following the same approach from Eq. (15), the $(1-\tau)100\%$ upper credible bound for θ can be obtained by solving the following equation for I_U:

$$\Gamma\left(a+m,\, (b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n})\,I_U\right) = \frac{\tau}{2}\Gamma(m+a). \tag{17}$$

The nonlinear Eqs. (16) and (17) prevents analytical solution, requiring appropriate numerical techniques to compute the values of I_L and I_U, respectively.

To find the $(1-\tau)100\%$ credible interval of the reliability function $r(t)$ we use the following algorithm:

Algorithm 1

- Step 1: Generate θ from $\pi(\theta|\mathbf{x})$ to get a sample of size M, $\{\theta_1, \theta_2, \ldots, \theta_M\}$.
- Step 2: Evaluate $r(t) = e^{-\theta t}$ for each θ_i, $i = 1,2,\ldots,M$ to get $r_1(t), r_2(t), \ldots, r_M(t)$.
- Step 3: Order $r_1(t), r_2(t), \ldots, r_M(t)$ to get the order statistics $r_{(1)}(t), r_{(2)}(t), \ldots, r_{(M)}(t)$.
- Step 4: The $(1-\tau)100\%$ credible interval of the reliability function $r(t)$ is given by $[r_{[\frac{M\tau}{2}]}(t), r_{[M(1-\frac{\tau}{2})]}(t)]$, where $[x]$ represents the greatest integer less than or equal to x.

4. Simulation study

This section presents our main empirical contribution: a systematic comparison of the estimation methods described above. We emphasize that this is a comparative study examining the finite-sample behavior of different statistical approaches, each with its own theoretical foundation and practical considerations. We do not claim that one paradigm is inherently superior to another, as maximum likelihood and Bayesian methods represent different philosophical approaches to statistical inference.

Our objective is to understand how these methods perform under various scenarios, which can provide practical guidance for researchers. The comparison focuses on finite-sample properties rather than establishing theoretical dominance.

Using simulated progressive Type II censored data under different censoring schemes, this research explores and assesses the performance and characteristics of maximum likelihood estimators and Bayes estimators. The following censoring schemes are considered:

- Case 1: $(n = 30, m = 10, r_1 = \ldots = r_4 = 5, r_5 = \ldots = r_{10} = 0)$
- Case 2: $(n = 30, m = 15, r_1 = \ldots = r_6 = 0, r_7 = r_8 = r_9 = 5, r_{10} = \ldots = r_{15} = 0)$
- Case 3: $(n = 30, m = 20, r_1 = \ldots = r_{18} = 0, r_{19} = r_{20} = 5)$
- Case 4: $(n = 40, m = 10, r_1 = \ldots = r_{10} = 3)$
- Case 5: $(n = 40, m = 20, r_1 = \ldots = r_{10} = 2, r_{11} = \ldots = r_{20} = 0)$

For the simulation study, we generated progressive type II censored samples from an exponential distribution with parameter $\theta = 2$ using the technique given by Balakrishnan and Aggarwala (2000). In the Bayesian approach, we have computed the Bayesian estimates for the exponential parameter θ and reliability function with respect to both loss functions: squared error (L_{SE}) and Kullback-Leibler (L_{KL}). For conducting Bayesian analysis, the gamma density function with shape and scale parameters a and b, respectively, is assumed as a prior density of θ. To compute the different Bayes estimates under the two considered loss functions SELF and KELF, we assume two priors:
Prior 0: $a = 0.01$, $b = 0.01$ (weakly informative prior)
Prior 1: $a = 1$, $b = 0.5$ (informative prior with prior mean $E[\theta] = 2$)
For comparing the performance of the estimators of θ and $r(t)$, we have computed the MSEs based on $M = 1000$ iterations, where

$$MSE = \frac{\sum_{i=1}^{M}(\theta_i - \hat{\theta}_E)^2}{M}, \tag{18}$$

where $\hat{\theta}_E$ stands for the point estimate computed by one of the considered method.

The results of the maximum likelihood estimates (MLEs) and the Bayes estimates (BEs) of θ and $r(t)$ under SELF and KELF are presented in Tables 1 and 2 for Prior 0 and Prior 1, respectively. The mean square errors (MSEs) are shown in parentheses. Table 3 compares the average lengths (A.L) of credible intervals for θ and $r(t)$ between Priors 0 and 1, and presents the coverage percentage (C.P) for all schemes considered.

Table 1. Results of MLEs and BEs when Prior 0 is applied under SELF and KELF ($\theta = 2$, $r(1) = \exp(-2) = 0.1353$)

n	m	Censoring Scheme	MLE (MSE)	BE under SELF (MSE)	BE under KELF (MSE)
30	10	$(4 \times 5, 6 \times 0)$	θ: 2.1218 (0.5019)	2.1189 (0.4975)	1.9072 (0.4002)
			$r(1)$: 0.1465 (0.0070)	0.1707 (0.0081)	0.1217 (0.0070)
30	15	$(6 \times 0, 3 \times 5, 6 \times 0)$	θ: 2.0849 (0.3338)	2.0832 (0.3318)	1.9444 (0.2861)
			$r(1)$: 0.1417 (0.0040)	0.1585 (0.0045)	0.1247 (0.0040)
30	20	$(18 \times 0, 2 \times 5)$	θ: 2.0787 (0.2791)	2.0775 (0.2778)	1.9736 (0.2460)
			$r(1)$: 0.1389 (0.0030)	0.1518 (0.0033)	0.1260 (0.0030)
40	10	(10×3)	θ: 2.1847 (0.5092)	2.1817 (0.5045)	1.9637 (0.3833)
			$r(1)$: 0.1379 (0.0069)	0.1622 (0.0075)	0.1132 (0.0072)
40	20	$(10 \times 2, 10 \times 0)$	θ: 2.1078 (0.2945)	2.1065 (0.2931)	2.0012 (0.2543)
			$r(1)$: 0.1365 (0.0032)	0.1493 (0.0035)	0.1237 (0.0033)

Table 2. Results of BEs when Prior 1 is applied under SELF and KELF ($\theta = 2$, $r(1) = exp(-2) = 0.1353$)

n	m	Censoring Scheme	BE under SELF (MSE)	BE under KELF (MSE)
30	10	$(4 \times 5, 6 \times 0)$	θ: 2.1101 (0.3662)	1.9183 (0.2993)
			$r(1)$: 0.1630 (0.0059)	0.1173 (0.0054)
30	15	$(6 \times 0, 3 \times 5, 6 \times 0)$	θ: 2.0591 (0.2665)	1.9304 (0.2360)
			$r(1)$: 0.1579 (0.0039)	0.1258 (0.0035)
30	20	$(18 \times 0, 2 \times 5)$	θ: 2.0770 (0.2050)	1.9781 (0.1810)
			$r(1)$: 0.1489 (0.0029)	0.1242 (0.0027)
40	10	(10×3)	θ: 2.1024 (0.4489)	1.9112 (0.3702)
			$r(1)$: 0.1682 (0.0072)	0.1232 (0.0062)
40	20	$(10 \times 2, 10 \times 0)$	θ: 2.0351 (0.2147)	1.9382 (0.1974)
			$r(1)$: 0.1551 (0.0033)	0.1304 (0.0028)

Table 3. Results of average lengths (A.L.) and coverage probabilities (C.P.)

			Prior 0		Prior 1	
n	m	Censoring Scheme	A.L.	C.P.	A.L.	C.P.
30	10	$(4\times 5, 6\times 0)$	θ: 2.91	0.94	2.47	0.96
			$r(1)$: 0.31	0.94	0.30	0.94
30	15	$(6\times 0, 3\times 5, 6\times 0)$	θ: 2.09	0.96	2.01	0.95
			$r(1)$: 0.27	0.96	0.26	0.95
30	20	$(18\times 0, 2\times 5)$	θ: 1.81	0.96	1.76	0.96
			$r(1)$: 0.23	0.96	0.22	0.95
40	10	(10×3)	θ: 2.67	0.95	2.46	0.94
			$r(1)$: 0.32	0.95	0.31	0.93
40	20	$(10\times 2, 10\times 0)$	θ: 1.80	0.93	1.73	0.94
			$r(1)$: 0.24	0.93	0.23	0.94

The following interpretations and comments can be obtained from these tables:

(1) Table 1 shows that increasing m improves the performance of MLEs of θ and $r(t)$ in terms of biases and mean square errors (MSEs), which is consistent with asymptotic theory.

(2) Tables 1 and 2 show that the Bayesian estimates of the parameter θ, derived from both Prior 0 and Prior 1 while employing the two loss functions SELF and KELF, show different characteristics. Under the weakly informative Prior 0, the Bayesian estimates under SELF are identical to the MLEs, as expected. The Kullback-Leibler loss function produces estimates that are generally closer to the true value in terms of bias. The performance differences between methods become less pronounced as the sample size increases, which is consistent with asymptotic theory.

(3) Table 2 indicates that when using the informative Prior 1, both Bayesian methods show improved performance compared to the weakly informative prior, demonstrating the value of accurate prior information. The Kullback-Leibler loss function continues to show competitive performance relative to squared error loss.

(4) Table 3 shows that the credible intervals behave as expected: average lengths decrease as sample size increases, and coverage probabilities are close to the nominal 95% level. The informative prior generally produces shorter intervals while maintaining appropriate coverage.

These results illustrate the practical implications of different methodological choices rather than establishing the superiority of one approach over another. Each method has its place depending on the availability of prior information and the specific loss structure of the problem.

Real data example

The dataset considered by Lawless (1982), as presented in Table 1.1 on page 3, represents the duration (measured in minutes) until the breakdown of an insulating fluid situated between electrodes at a voltage of 34 kV. The complete dataset, comprising 19 recorded breakdown times, properly ordered, is: 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89.

In order to ascertain the suitability of this dataset for the exponential distribution, we employed the Kolmogorov-Smirnov (KS) test. The rate parameter of the exponential distribution was estimated using the maximum likelihood estimation (MLE) method from the entire dataset ($n = 19$), yielding $\hat{\theta} = 1/\bar{x} = 0.06966$, where $\bar{x}$ is the sample mean. The KS test statistic was calculated using the standard definition $D_n = \max|F(x) - \hat{F}(x)|$, where $F(x)$ is the empirical cumulative distribution function and $\hat{F}(x)$ is the theoretical cumulative distribution function of the exponential distribution with the estimated parameter $\hat{\theta} = 0.06966$. The test statistic ($D_n = 0.2463$) and the corresponding p-value (0.1993) were computed using Minitab Statistical Software (Version 15) based on the asymptotic Kolmogorov distribution formula:

$$P(D_n > \frac{z}{\sqrt{n}}) = 2\sum_{k=1}^{\infty}(-1)^{k-1}\exp(-2k^2z^2)$$

where D_n is the Kolmogorov-Smirnov statistic, $\frac{z}{\sqrt{n}} = 0.2463$, and $n = 19$. Graphical diagnostics including empirical and theoretical cumulative distribution functions (CDFs) and the quantile-quantile (Q-Q) plot were created using R software with the `ggplot2` package and are presented in Figures 1 and 2. With p-value $= 0.1993 > 0.05$, we fail to reject the null hypothesis at the $\alpha = 0.05$ significance level, indicating that the exponential distribution is appropriate for modeling this dataset.

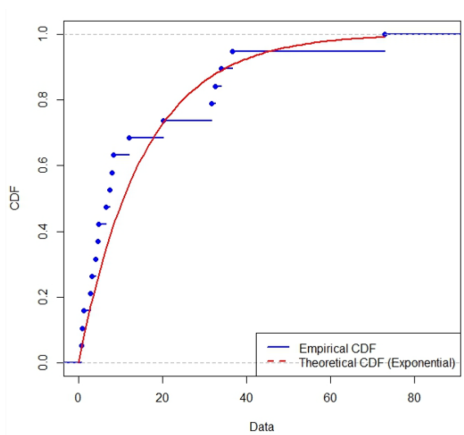

Figure 1. Empirical and theoretical cumulative distribution functions for the breakdown voltage data. The close agreement between the empirical CDF (step function) and theoretical exponential CDF (smooth curve) supports the exponential distribution assumption

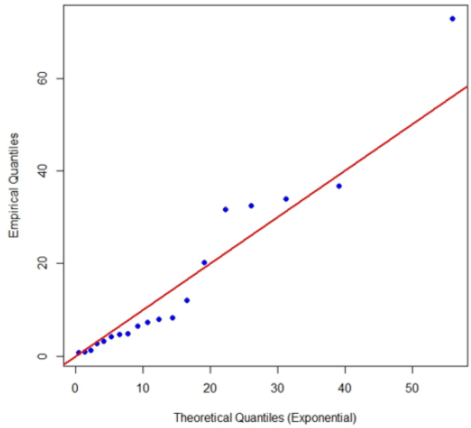

Figure 2. Q-Q Plot: Empirical vs. Theoretical Quantiles (Exponential). The points approximately follow the diagonal line, indicating good fit to the exponential distribution, with some deviation in the upper tail as commonly observed in reliability data

To demonstrate the methods developed in this study, we analyze this dataset and extract $m = 12$ progressive type II sample from this data using the scheme:

i	1	2	3	4	5	6	7	8	9	10	11	12
r_i	1	0	0	2	0	0	1	0	1	0	0	2
$x_{i:m:n}$	0.19	0.96	1.31	2.75	4.67	4.85	6.50	8.01	8.27	31.75	32.52	33.91

This progressive censoring scheme works as follows: at the first failure (0.19), we remove 1 additional item (0.78); at the fourth failure (2.78), we remove 2 items (3.16, 4.15); at the seventh failure (6.50), we remove 1 item (7.35); at the ninth failure (8.27), we remove 1 item (12.06); and at the final failure (33.91), we remove the remaining 2 items (36.71, 72.89). This accounts for all 19 original observations: 12 observed failures + 7 censored items = 19 total.

For this dataset, we applied our methods and obtained the maximum likelihood estimates of θ and $r(t)$. All computations were performed using Mathematica 12.0 software. The MLE was computed using Equation (5) from Section 2. The Bayesian estimates were obtained using Equations (10) and (11) from Section 3 under both squared error and Kullback-Leibler loss functions. The credible intervals were computed numerically by solving Equations (16) and (17) using Mathematica's `FindRoot` function with precision settings `WorkingPrecision`→16 and `MaxIterations`→1000. The estimates were determined to be approximately 0.0536 and 0.9479, respectively, and 95% confidence intervals for θ and $r(t)$ were determined to be respectively (0.0233, 0.0839) and (0.9191, 0.9766). Since $h(t) = \theta$ for the exponential distribution, $\hat{h}(t)_{MLE} = 0.0536$. Furthermore, as illustrated in Table 4, the Bayes estimates along with the corresponding 95% credible intervals for both Priors 0 and 1 are presented. The estimates from different methods show reasonable variation, reflecting the different loss functions and prior specifications and fall within the established 95% credible intervals. When Prior 1 is employed, the estimates reflect the influence of the informative prior, showing how prior knowledge affects the results.

Table 4. Estimation results for the real data with 95% credible intervals ($(\theta = 0.0536, r(1) = exp(-0.0536) = 0.9478)$)

Scheme: n=19 ,	m=12	$(1, 2\times 0, 2, 2\times 0, 1, 0, 1, 2\times 0, 2)$		
		SELF	*KELF*	95% credible interval
Prior 0	θ	0.0536	0.0492	$(0.0277, 0.0879)$
	$r(t)$	0.9479	0.9476	$(0.9158, 0.9727)$
Prior 1	θ	0.0579	0.0535	$(0.0308, 0.0934)$
	$r(t)$	0.9438	0.9436	$(0.9109, 0.9696)$

The results demonstrate the practical application of the different estimation methods. The MLE and Bayesian estimates under weakly informative priors are quite similar, while the informative prior shows its influence on the estimates. The Kullback-Leibler loss function consistently produces somewhat different estimates, reflecting its different optimization criterion compared to squared error loss.

Following Lawless (1982)'s framework for Breakdown Voltage Data, Figures 3 and 4 visually compare the performance of all estimation methods.

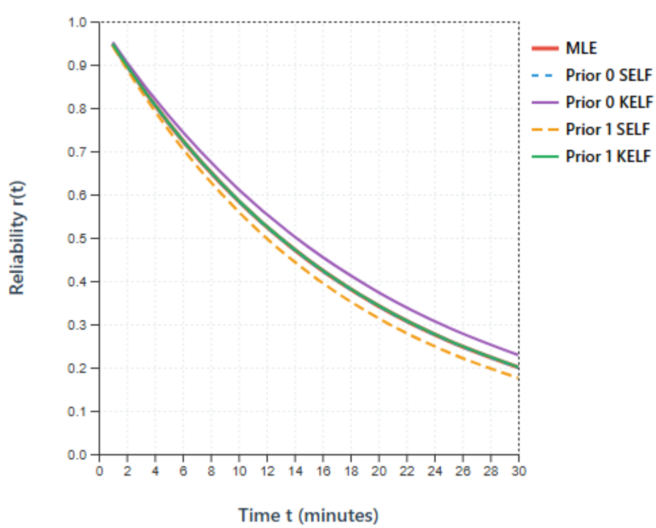

Figure 3. Reliability Function $r(t)$ Estimates under Different Loss Functions and Priors

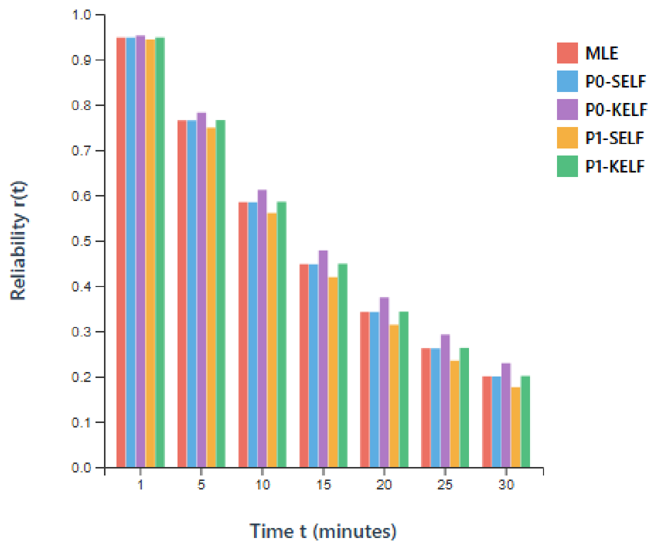

Figure 4. Reliability Estimates at Selected Time Points by Estimation Method

5. Conclusion

This paper provides a comparative analysis of frequentist and Bayesian estimation methods for the parameter, reliability and hazard functions of the exponential distribution using progressive type II censored data. The theoretical methods presented are well established in the statistical literature; our contribution lies in the systematic empirical comparison of these approaches.

The maximum likelihood estimate and Bayes estimates of the parameter, reliability and hazard functions have been computed using standard procedures implemented via "Mathematica 12" software. Two loss functions have been considered: squared error and Kullback-Leibler for Bayesian computation under two priors: weakly informative and informative. The performance of all estimators has been assessed through mean square errors.

The key contributions include:

- under weakly informative priors, Bayesian estimates with squared error loss and maximum likelihood estimators show similar performance, confirming theoretical expectations about their asymptotic equivalence,
- the Kullback-Leibler loss function provides a meaningful alternative with different optimization characteristics,
- informative priors can improve estimation performance when they accurately reflect prior knowledge,
- the choice between methods should be guided by the specific context, available prior information, and loss structure rather than claims of universal superiority,

- proper acknowledgment that $h(t) = \theta$ for exponential distributions, avoiding redundant calculations.

The analysis of real reliability data illustrates the practical application of these methods and shows reasonable agreement among different approaches.

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